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# BRIDGELAND STABILITY CONDITIONS ON THREEFOLDS II: AN APPLICATION TO FUJITA'S CONJECTURE

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**ABSTRACT.** We apply a conjectured inequality on third chern classes of stable two-term complexes on threefolds to Fujita's conjecture. More precisely, the inequality is shown to imply a Reider-type theorem in dimension three which in turn implies that  $K_X + 6L$  is very ample when  $L$  is ample, and that  $5L$  is very ample when  $K_X$  is trivial.

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## 1. INTRODUCTION

A Bogomolov-Gieseker-type inequality on Chern classes of “tilt-stable” objects in the derived category of a threefold was conjectured in [BMT11] in the context of constructing Bridgeland stability conditions. In this paper, we show how the same inequality would allow one to extend Reider's stable-vector bundle technique ([Rei88]) from surfaces to threefolds, and in particular to obtain Fujita's conjecture in the threefold case. This follows a line of reasoning that was suggested in [AB11].

While we use the setup of tilt-stability from [BMT11], this paper is intended to be self-contained, and to be readable by birational geometers with a passing familiarity with derived categories.

Tilt-stability depends on two numerical parameters: an ample class  $\omega \in \mathrm{NS}_{\mathbb{Q}}(X)$  and an arbitrary class  $B \in \mathrm{NS}_{\mathbb{Q}}(X)$ . It is a notion of stability on a particular abelian category,  $\mathcal{B}_{\omega,B}$ , of two-term complexes in  $D^b(X)$ , and codimension three Chern classes of stable

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objects  $E$  in this category (and not stable vector bundles) are conjectured to satisfy a Bogomolov-Gieseker inequality in Conjecture 2.3. Assuming this conjecture, we prove the following Reider-type theorem for threefolds:

**Theorem 4.1.** *Let  $X$  be a smooth projective threefold over  $\mathbb{C}$ , and let  $L$  be an ample line bundle on  $X$  such that Conjecture 2.3 holds when  $B$  and  $\omega$  are scalar multiples of  $L$ . Fix a positive integer  $\alpha$ , and assume that  $L$  satisfies the following conditions:*

- (A)  $L^3 > 49\alpha$ ;
- (B)  $L^2.D \geq 7\alpha$ , for all integral divisor classes  $D$  with  $L^2.D > 0$  and  $L.D^2 < \alpha$ ;
- (C)  $L.C \geq 3\alpha$ , for all curves  $C$ .

*Then  $H^1(X, K_X \otimes L \otimes I_Z) = 0$  for any zero-dimensional subscheme  $Z \subset X$  of length  $\alpha$ .*

Theorem 4.1 would give an effective numerical criterion for an adjoint line bundle to be globally generated ( $\alpha = 1$ ) or very ample ( $\alpha = 2$ ):

**Corollary 1.1** (Fujita's Conjecture). *Let  $L$  be an ample line bundle on a smooth projective threefold  $X$ . Assume Conjecture 2.3 holds for  $\omega$  and  $B$  as above. Then:*

- (a)  $K_X \otimes L^{\otimes m}$  is globally generated for  $m \geq 4$ . Moreover, if  $L^3 \geq 2$ , then  $K_X \otimes L^{\otimes 3}$  is also globally generated.
- (b)  $K_X \otimes L^{\otimes m}$  is very ample for  $m \geq 6$ .

In Proposition 4.2, we also show (assuming the conjecture) that  $K_X \otimes L^5$  is very ample as long as its restriction to special degree one curves is very ample. As a consequence,  $K_X \otimes L^5$  is very ample when  $K_X$  is trivial, or, more generally, when  $K_X.C$  is even for all curves  $C \subset X$ .

Ein and Lazarsfeld proved that  $K_X \otimes L^{\otimes 4}$  is globally generated [EL93]. In the case  $L^3 \geq 2$ , Fujita, Kawamata, and Helmke proved that  $K_X \otimes L^{\otimes 3}$  is globally generated as well [Fuj93, Kaw97, Hel97]. In fact, in Proposition 4.4, we show that these results conversely give some evidence for Conjecture 2.3. Case (b) in Corollary 1.1 instead is not known in general; but also note that the strongest form of Fujita's conjecture predicts that  $K_X \otimes L^{\otimes 5}$  is already very ample. For further references, we refer to [Laz04, Section 10.4]. Notice that the bounds in Theorem 4.1 are very similar to those in [Fuj93] when  $\alpha = 1$  (see also [Kaw97, Hel97]) and, when  $\alpha = 2$  and  $Z$  consists of two distinct points, to those in [Fuj94].

**Approach.** We explain our approach, which was outlined in [AB11, Section 5], but can now be made precise using the strong Bogomolov-Gieseker conjecture of [BMT11]. It is closer to Reider's original approach [Rei88] for surfaces via stability of sheaves (generalized to threefolds by extending it to derived categories), than to the Ein-Lazarsfeld-Kawamata approach mentioned above, via vanishing theorems.

Let us give first a brief recall on Reider's method for proving Fujita's Conjecture in the case of  $X$  being a surface. By Serre duality, an adjoint linear system  $K_X \otimes L$  is very ample

if and only if  $\text{Ext}^1(L \otimes I_Z, \mathcal{O}_X) = H^1(X, K_X \otimes L \otimes I_Z)^\vee = 0$ , for all zero-dimensional subscheme  $Z \subset X$  of length one or two. If this group was non-zero, we would get a rank 2 torsion-free sheaf  $E$  as the non-trivial extension  $\mathcal{O}_X \hookrightarrow E \rightarrow L \otimes I_Z$ . Reider's idea is to consider the slope-stability of  $E$ . If  $E$  is stable, then the classical Bogomolov-Gieseker inequality gives a bound on the degree  $L^2$  of  $L$  in terms of the length of  $Z$ . If  $E$  is not stable, then the destabilizing subsheaf gives a curve of bounded degree with respect to  $L$ . Hence, if we assume that  $L$  satisfies inequalities similar to (A) and (C), we would get a contradiction.

We generalize this approach to threefolds as follows. We suppose the conclusion of Theorem 4.1 is false. Then by Serre duality,

$$0 \neq \text{Ext}^2(L \otimes I_Z, \mathcal{O}_X) = \text{Ext}^1(L \otimes I_Z, \mathcal{O}_X[1]).$$

For appropriate choices of  $\omega$  and  $B$ , both  $L \otimes I_Z$  and  $\mathcal{O}_X[1]$  are objects in the abelian category  $\mathcal{B}_{\omega, B}$ , and thus this extension class corresponds to another object  $E$  of  $\mathcal{B}_{\omega, B}$ . In Section 3.1, we will show that for  $\omega \rightarrow 0$ , the complex  $E$  violates the inequality of Conjecture 2.3, thus it must become unstable. We show in Section 3.2 that the Chern classes of a destabilizing subobject give a contradiction to Assumptions (A) and (B) of the Theorem unless it is of the form  $L \otimes I_C$ , where  $I_C$  is the ideal sheaf of a curve containing  $Z$ . In Section 4, we apply our conjecture and Assumption (C) to this remaining case and deduce Theorem 4.1.

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**Notation and Convention.** Throughout the paper,  $X$  will be a smooth projective threefold defined over  $\mathbb{C}$  and  $\text{D}^b(X)$  its bounded derived category of coherent sheaves. Given a line bundle  $L$  on  $X$ , we will denote by  $\mathbb{D}_L: \text{D}^b(X) \rightarrow \text{D}^b(X)$  the following local dualizing functor on its derived category:

$$\mathbb{D}_L(\_) := (\_)^\vee[1] \otimes L = \mathbf{R}\mathcal{H}om(\_, L[1]).$$

We identify a line bundle  $L$  with its first Chern class  $c_1(L)$ , and write  $K_X$  for the canonical line bundle. While  $L^{\otimes m}$  denotes the tensor powers of the line bundle,  $L^k$  denotes the intersection product of its first Chern class.

## 2. SETUP

In this section, we briefly recall the notion of “tilt-stability” defined in [BMT11, Section 3] and its most important properties.

Let  $X$  be a smooth projective threefold, and let  $\omega, B \in \text{NS}_{\mathbb{Q}}(X)$  be rational numerical divisor classes such that  $\omega$  is ample. We use  $\omega, B$  to define a slope function  $\mu_{\omega, B}$  for coherent sheaves on  $X$  as follows: For torsion sheaves  $E$ , we set  $\mu_{\omega, B}(E) = +\infty$ , otherwise

$$\mu_{\omega, B}(E) = \frac{\omega^2 \text{ch}_1^B(E)}{\omega^3 \text{ch}_0^B(E)} = \frac{\omega^2 \text{ch}_1(E)}{\omega^3 \text{ch}_0^B(E)} - \frac{\omega^2 B}{\omega^3}$$

where  $\text{ch}^B(E) = e^{-B} \text{ch}(E)$  denotes the Chern character twisted by  $B$  (explicitly,  $\text{ch}_0^B = \text{rk}$ ,  $\text{ch}_1^B = c_1 - B \text{rk}$ , etc.).

A coherent sheaf  $E$  is slope-(semi)stable (or  $\mu_{\omega, B}$ -(semi)stable) if, for all subsheaves  $F \hookrightarrow E$ , we have

$$\mu_{\omega, B}(F) < (\leq) \mu_{\omega, B}(E/F).$$

Due to the existence of Harder-Narasimhan filtrations (HN-filtrations, for short) with respect to slope-stability, there exists a “torsion pair”  $(\mathcal{T}_{\omega, B}, \mathcal{F}_{\omega, B})$  defined as follows:

$$\begin{aligned} \mathcal{T}_{\omega, B} &= \{E \in \text{Coh } X : \text{any quotient } E \twoheadrightarrow G \text{ satisfies } \mu_{\omega, B}(G) > 0\} \\ \mathcal{F}_{\omega, B} &= \{E \in \text{Coh } X : \text{any subsheaf } F \hookrightarrow E \text{ satisfies } \mu_{\omega, B}(F) \leq 0\} \end{aligned}$$

Equivalently,  $\mathcal{T}_{\omega, B}$  and  $\mathcal{F}_{\omega, B}$  are the extension-closed subcategories of  $\text{Coh } X$  generated by slope-stable sheaves of positive or non-positive slope, respectively.

**Definition 2.1.** We let  $\mathcal{B}_{\omega, B} \subset \text{D}^b(X)$  be the extension-closure

$$\mathcal{B}_{\omega, B} = \langle \mathcal{T}_{\omega, B}, \mathcal{F}_{\omega, B}[1] \rangle.$$

More explicitly,  $\mathcal{B}_{\omega, B}$  is the subcategory of two-term complexes  $E: E^{-1} \xrightarrow{d} E^0$  with  $H^{-1}(E) = \ker d \in \mathcal{F}_{\omega, B}$  and  $H^0(E) = \text{cok } d \in \mathcal{T}_{\omega, B}$ . We can characterize isomorphism classes of objects in  $\mathcal{B}_{\omega, B}$  by extension classes: to give an object  $E \in \mathcal{B}_{\omega, B}$  is equivalent to giving  $T \in \mathcal{T}_{\omega, B}$ ,  $F \in \mathcal{F}_{\omega, B}$ , and a class  $\xi \in \text{Ext}_X^2(T, F)$ .

By the general theory of torsion pairs and tilting [HRO96],  $\mathcal{B}_{\omega, B}$  is the heart of a bounded t-structure on  $\text{D}^b(X)$ . For the most part, we only need that  $\mathcal{B}_{\omega, B}$  is an abelian category: Exact sequences in  $\mathcal{B}_{\omega, B}$  are given by exact triangles in  $\text{D}^b(X)$ . For any such exact sequence

$$0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$$

in  $\mathcal{B}_{\omega, B}$ , we have a long exact sequence in  $\text{Coh } X$ :

$$\begin{aligned} 0 \rightarrow H^{-1}(E) \rightarrow H^{-1}(F) \rightarrow H^{-1}(G) \rightarrow \\ \rightarrow H^0(E) \rightarrow H^0(F) \rightarrow H^0(G) \rightarrow 0. \end{aligned}$$

Using the classical Bogomolov-Gieseker inequality and Hodge Index theorem, we defined the following slope function on  $\mathcal{B}_{\omega,B}$ : We set  $\nu_{\omega,B}(E) = +\infty$  when  $\omega^2 \text{ch}_1^B(E) = 0$ , and otherwise

$$(1) \quad \nu_{\omega,B}(E) = \frac{\omega \text{ch}_2^B(E) - \frac{1}{6}\omega^3 \text{ch}_0^B(E)}{\omega^2 \text{ch}_1^B(E)}.$$

We showed that this is a slope function, in the sense that it satisfies the weak see-saw property for short exact sequences in  $\mathcal{B}_{\omega,B}$ : for any subobject  $F \hookrightarrow E$ , we have  $\nu_{\omega,B}(F) \leq \nu_{\omega,B}(E) \leq \nu_{\omega,B}(E/F)$  or  $\nu_{\omega,B}(F) \geq \nu_{\omega,B}(E) \geq \nu_{\omega,B}(E/F)$ .

**Definition 2.2.** An object  $E \in \mathcal{B}_{\omega,B}$  is “tilt-(semi)stable” if, for all non-trivial subobjects  $F \hookrightarrow E$ , we have

$$\nu_{\omega,B}(F) < (\leq) \nu_{\omega,B}(E/F).$$

Motivated by the case of torsion sheaves ([BMT11, Proposition 7.1.1]), by projectively flat vector bundles ([BMT11, Proposition 7.4.2]), and the case of  $X = \mathbb{P}^3$  ([BMT11, Theorem 8.2.1] and [Mac12]), we stated the following conjecture:

**Conjecture 2.3** ([BMT11, Conjecture 1.3.1]). *For any  $\nu_{\omega,B}$ -semistable object  $E \in \mathcal{B}_{\omega,B}$  satisfying  $\nu_{\omega,B}(E) = 0$ , we have the following inequality*

$$(2) \quad \text{ch}_3^B(E) \leq \frac{\omega^2}{18} \text{ch}_1^B(E).$$

Conjecture 2.3 is analogous to the classical Bogomolov-Gieseker inequality, which can be formulated as follows: For any  $\mu_{\omega,B}$ -semistable sheaf  $E$  satisfying  $\mu_{\omega,B}(E) = 0$ , we have  $\omega \text{ch}_2^B(E) \leq 0$ .

The original motivation for Conjecture 2.3 is to construct examples of Bridgeland stability conditions on  $\text{D}^b(X)$ . While any linear inequality of the form (2) would be sufficient to this end, the constant  $\frac{1}{18}$  in equation (2) is chosen so that, if  $\omega$  and  $B$  are proportional to the first Chern class of an ample line bundle  $L$ , the inequality is an equality for tensor power  $L^{\otimes n}$  of  $L$ . More generally, it is an equality when  $E$  is a slope-stable vector bundles  $E$  whose discriminant  $\Delta = (\text{ch}_1^B)^2 - 2 \text{ch}_0^B \text{ch}_2^B$  satisfies  $\omega \Delta(E) = 0$ , and for which  $\text{ch}_1^B(E)$  is proportional to  $L$ . Such vector bundles have a projectively flat connection, and are examples of tilt-stable objects:

**Proposition 2.4** ([BMT11, Proposition 7.4.1]). *Let  $L$  be an ample line bundle, and assume that both  $\omega$  and  $B$  are proportional to  $L$ . Then any slope-stable vector bundle  $E$ , with  $\omega \Delta(E) = 0$  and for which  $\text{ch}_1^B(E)$  is proportional to  $L$ , is also tilt-stable with respect to  $\nu_{\omega,B}$ .*

The proof is essentially the same as for line bundles  $L^{\otimes n}$  in [AB11, Proposition 3.6].

By assuming Conjecture 2.3, we can also show conversely: if an object in  $\mathcal{B}_{\omega,B}$  is tilt-stable and the inequality in Conjecture 2.3 is an equality, then it must have trivial

discriminant. We first recall that, based on Bridgeland's deformation theorem in [Bri07], we also showed the existence of a continuous family of stability conditions depending on real classes  $\omega, B$ :

**Proposition 2.5** ([BMT11, Corollary 3.3.3]). *Let  $U \subset \text{NS}_{\mathbb{R}}(X) \times \text{NS}_{\mathbb{R}}(X)$  be the subset of pairs of real classes  $(\omega, B)$  for which  $\omega$  is ample. There exists a notion of “tilt-stability” for every  $(\omega, B) \in U$ . For every object  $E$ , the set of  $(\omega, B)$  for which  $E$  is  $\nu_{\omega, B}$ -stable defines an open subset of  $U$ .*

By using Proposition 2.5, we can then prove the following.

**Proposition 2.6.** *Let  $L$  be an ample line bundle, and assume that both  $\omega$  and  $B$  are proportional to  $L$ . Assume also that Conjecture 2.3 holds for such  $B$  and  $\omega$ . Let  $E \in \mathcal{B}_{\omega, B}$  be a  $\nu_{\omega, B}$ -stable object, with  $\text{ch}_0(E) \neq 0$  and  $\text{ch}_1(E)$  proportional to  $L$ , and satisfying:*

$$\frac{\omega^3}{6} \text{ch}_0(E) = \omega \text{ch}_2^B(E) \quad \text{and} \quad \text{ch}_3^B(E) = \frac{\omega^2}{18} \text{ch}_1^B(E).$$

Then  $\omega \cdot \Delta(E) = 0$ .

*Proof.* Write  $d = L^3$ ,  $B = b_0 L$ ,  $\omega = T_0 L$  and  $\text{ch}_0(E) = r$ . The idea for the proof is that, since stability is an open property, we can deform  $b = b_0$  and  $T = T_0$ , as a function  $T = T(b)$  of  $b$ , slightly such that  $E$  is still  $\nu_{T(b)L, bL}$ -stable with  $\nu_{T(b)L, bL}(E) = 0$ ; then we apply Conjecture 2.3 for the pairs  $\omega = T(b)L$ ,  $B = bL$  depending on  $b$ .

Evidently,  $\nu_{T(b)L, bL}(E) = 0$  is equivalent to

$$T^2 = \frac{6}{rd} L \cdot \text{ch}_2^{bL}(E)$$

Since  $T_0 > 0$ , and since the equation is satisfied for  $T = T_0$  and  $b = b_0$ , the equation defines a function  $T = T(b)$  for  $b$  nearby  $b_0$ .

It is immediate to check from the definition that the chain rule

$$(3) \quad \frac{\partial}{\partial b} \text{ch}_i^{bL}(E) = -L \text{ch}_{i-1}^{bL}(E)$$

holds for  $i = 1, \dots, 3$ .

Consider

$$f(b) = \text{ch}_3^{bL}(E) - \frac{(T(b)L)^2}{18} \cdot \text{ch}_1^{bL}(E) = \text{ch}_3^{bL}(E) - \frac{1}{3rd} L \cdot \text{ch}_2^{bL}(E) \cdot L^2 \cdot \text{ch}_1^{bL}(E)$$

as a function of  $b$  in some neighborhood of  $b_0 \in \mathbb{R}$ . By Proposition 2.5 and Conjecture 2.3, we have  $f(b) \leq 0$  for  $b$  close to  $b_0$ , and by assumption  $f(b_0) = 0$ ; therefore  $f'(b_0) = 0$ . Using equation (3), we obtain

$$\begin{aligned} f'(b) &= -L \cdot \text{ch}_2^{bL}(E) + \frac{1}{3rd} ((L^2 \cdot \text{ch}_1^{bL})^2 + L \cdot \text{ch}_2^{bL}(E) \cdot rd) \\ &= \frac{1}{3r} (L \cdot (\text{ch}_1^{bL}(E))^2 - 2L \cdot \text{ch}_2^{bL}(E)r) = \frac{1}{3r} L \cdot \Delta(E). \end{aligned}$$

(Note that we used  $(L^2 \cdot \text{ch}_1^{bL})^2 = L^3 \cdot L \cdot (\text{ch}_1^{bL})^2$ , which holds because  $\text{ch}_1^{bL}(E)$  is proportional to  $L$ .) This proves the claim.  $\square$

Finally, based on an alternate construction of tilt-stability, we also showed that it behaves well with respect to the dualizing functor  $\mathbb{D}_L(\_) = \mathbf{R}\mathcal{H}om(\_, L[1])$  for every line bundle  $L$ . For this purpose, we fix  $B = \frac{L}{2}$ :

**Proposition 2.7.** *Let  $F \in \mathcal{B}_{\omega, \frac{L}{2}}$  be an object with  $\nu_{\omega, \frac{L}{2}}(A) < +\infty$  for every subobject  $A \subset F$ . Then there is an exact triangle  $\tilde{F} \rightarrow \mathbb{D}_L(F) \rightarrow T_0[-1]$  where  $T_0$  is a zero-dimensional torsion sheaf and  $\tilde{F}$  an object of  $\mathcal{B}_{\omega, \frac{L}{2}}$  with  $\nu_{\omega, \frac{L}{2}}(\tilde{F}) = -\nu_{\omega, \frac{L}{2}}(F)$ . The object  $\tilde{F}$  is  $\nu_{\omega, \frac{L}{2}}$ -semistable if and only if  $F$  is  $\nu_{\omega, \frac{L}{2}}$ -semistable.*

*Proof.* Since  $\mathbb{D}_L(\_)$  can be written as the composition  $\_ \otimes L \circ \mathbb{D}(\_)$ , this follows from [BMT11, Proposition 5.1.3] and the fact that tensoring with  $L$  corresponds to replacing  $B$  with  $B - L$ .  $\square$

### 3. REDUCTION TO CURVES

In this section, we use Assumptions (A) and (B) of Theorem 4.1 to show that the non-vanishing of  $H^1(X, K_X \otimes L \otimes I_Z)$  implies the existence of special low-degree curves on  $X$ . The approach, explained in the introduction, involves studying the tilt-stability of a certain object  $E$  in the category  $\mathcal{B}$  constructed in the previous section.

**3.1. Bogomolov-Gieseker inequalities and stability.** We will use Conjecture 2.3 in the case where  $L$  is an ample line bundle on  $X$ ,  $\omega = TL$  for some  $T > 0$ , and  $B = \frac{L}{2}$ . The abelian category  $\mathcal{B} := \mathcal{B}_{TL, \frac{L}{2}}$  is independent of  $T$ .

To simplify notation, we will rescale the slope function: set  $t = \frac{T^2}{6}$  and write  $\nu_t$  for

$$(4) \quad \nu_t(\_) = T \cdot \nu_{TL, \frac{L}{2}}(\_) = \frac{L \cdot \text{ch}_2^{L/2}(\_) - td \text{ch}_0^{L/2}(\_)}{L^2 \cdot \text{ch}_1^{L/2}(\_)},$$

where  $d := L^3$ . Then the inequality of Conjecture 2.3 states that, for every  $\nu_t$ -stable object  $E$ , we have

$$(5) \quad \text{ch}_3^{L/2}(E) \leq \frac{t}{3} L^2 \cdot \text{ch}_1^{L/2}(E) \quad \text{if} \quad L \cdot \text{ch}_2^{L/2}(E) = dt \text{ch}_0^{L/2}(E).$$

Let  $Z \subset X$  be a zero-dimensional subscheme of length  $\alpha$ . Following [AB11], observe that if  $H^1(X, K_X \otimes L \otimes I_Z) \neq 0$ , then by Serre duality, we also have  $\text{Ext}^2(L \otimes I_Z, \mathcal{O}_X) \neq 0$ . Any non-zero element  $\xi \in \text{Ext}^2(L \otimes I_Z, \mathcal{O}_X)$  gives a non-trivial exact triangle in  $\text{D}^b(X)$

$$(6) \quad \mathcal{O}_X[1] \rightarrow E = E_\xi \rightarrow L \otimes I_Z \xrightarrow{\xi} \mathcal{O}_X[2].$$

We will show that  $E$  is  $\nu_t$ -semistable for  $t = \frac{1}{8}$ ; its Chern classes invalidate the inequality of Conjecture 2.3 for  $t \ll 1$ , and thus it must become unstable for  $t < t_0$  and some



$t_0 \in (0, \frac{1}{8}]$ ; finally, we will show that the Chern classes of its destabilizing factor would give special curves or divisors on  $X$ .

**Proposition 3.1.** *Assume that  $H^1(X, K_X \otimes L \otimes I_Z) \neq 0$ , and let  $E$  be an extension as given by equation (6).*

(a)  $E \in \mathcal{B}$  and

$$\mathrm{ch}^{L/2}(E) = \left(0, L, 0, \frac{d}{24} - \alpha\right).$$

(b) If  $t > \frac{1}{8}$ , then (6) destabilizes  $E$  with respect to  $\nu_t$ .

(c) If  $t = \frac{1}{8}$ , then  $E$  is  $\nu_t$ -semistable.

(d) Assume Conjecture 2.3 and Assumption (A) of Theorem 4.1. Then  $E$  is not  $\nu_t$ -semistable for  $0 < t \ll 1$ ,

*Proof.* First of all, we have

$$\begin{aligned} \mathrm{ch}^{L/2}(\mathcal{O}_X) &= \left(1, -\frac{L}{2}, \frac{L^2}{8}, -\frac{L^3}{48}\right), \\ \mathrm{ch}^{L/2}(L \otimes I_Z) &= \left(1, \frac{L}{2}, \frac{L^2}{8}, \frac{L^3}{48} - \alpha\right). \end{aligned}$$

As  $\mathcal{O}_X$  and  $L \otimes I_Z$  are slope-stable, with  $\mu_{\omega, L/2}(\mathcal{O}_X) < 0$  and  $\mu_{\omega, L/2}(L \otimes I_Z) > 0$ , we have  $\mathcal{O}_X \in \mathcal{F}$  and  $L \otimes I_Z \in \mathcal{T}$ . By the definition of  $\mathcal{B}$ , it follows that  $\mathcal{O}_X[1]$ ,  $L \otimes I_Z$  and  $E$  are all objects of  $\mathcal{B}$ ; in particular, we have proved (a).

Moreover, we have

$$(7) \quad \nu_t(\mathcal{O}_X[1]) = 2 \left(t - \frac{1}{8}\right), \quad \nu_t(E) = 0$$

which immediately implies (b), since (6) is an exact sequence in  $\mathcal{B}$ .

To prove (c), simply observe that, by Proposition 2.4, both  $\mathcal{O}_X[1]$  and  $L$  are  $\nu_t$ -stable for all  $t > 0$ . Moreover, since  $\nu_t(L \otimes I_Z) = \nu_t(L)$ , any destabilizing subobject  $A \hookrightarrow L \otimes I_Z$  would also destabilize  $L$  via the composition  $A \hookrightarrow L \otimes I_Z \hookrightarrow L$  (which is an inclusion in  $\mathcal{B}$ ); thus  $L \otimes I_Z$  is also  $\nu_t$ -stable. For  $t = \frac{1}{8}$ , we have  $\nu_t(\mathcal{O}_X[1]) = \nu_t(L \otimes I_Z) = 0$ , and thus the extension (6) shows that  $E$  is  $\nu_t$ -semistable at  $t = \frac{1}{8}$ .

Finally, if  $E$  was  $\nu_t$ -semistable for all  $t \in (0, \frac{1}{8}]$ , then by our conjectural inequality (5) we would get

$$(8) \quad \frac{d}{24} - \alpha \leq \frac{t}{3}d$$

for all such  $t$ . Hence  $d \leq 24\alpha$ , in contradiction to Assumption (A).  $\square$

Notice that the previous proposition would answer Question 4 in [AB11]. Also observe that in part (d), instead of Assumption (A), already assuming  $d > 24\alpha$  would have been

enough. Similarly, instead of Conjecture 2.3, any linear inequality between  $\text{ch}_3^B$  and  $\text{ch}_1^B$  would have been sufficient.

In the following proposition, we will show that our situation is self-dual with respect to the local dualizing functor  $\mathbb{D}_L(\_) = \mathbf{R}\mathcal{H}om(\_, L[1])$ . As a preliminary, let us first note that we may make the following assumption:

(\*)  $H^1(X, K_X \otimes L \otimes I_{Z'}) = 0$  for all subschemes  $Z' \subsetneq Z$ , and  $H^1(X, K_X \otimes L \otimes I_Z) \cong \mathbb{C}$ .

Indeed, in order to show  $H^1(X, L \otimes I_Z \otimes K_X) = 0$ , we can proceed by induction on the length of  $Z$  (the case  $\alpha = 0$  is, of course, given by Kodaira vanishing).

**Proposition 3.2.** *If Assumption (\*) holds, and  $E$  is given by the unique non-trivial extension of the form (6), then  $E \cong \mathbb{D}_L(E)$ .*

*Proof.* Due to Assumption (\*), it is sufficient to show that  $\mathbb{D}_L(E)$  is again a non-trivial extension of the form (6). Applying the octahedral axiom to the composition  $\mathcal{O}_Z[-1] \rightarrow L \otimes I_Z \rightarrow \mathcal{O}_X[2]$ , and using the two exact triangles (6) and  $\mathcal{O}_Z[-1] \rightarrow L \otimes I_Z \rightarrow L$ , we obtain an exact triangle  $F \rightarrow E \rightarrow L$ , where  $F$  itself fits into an exact triangle

$$(9) \quad \mathcal{O}_X[1] \rightarrow F \rightarrow \mathcal{O}_Z[-1].$$

We claim that  $\text{Hom}(k(x)[-1], F) = 0$  for all skyscraper sheaves of points  $x \in X$ . Using the long exact sequence for  $\text{Hom}(k(x), \_)$  applied to (9), we see that this is equivalent to the non-vanishing of the composition

$$(10) \quad k(x)[-1] \rightarrow \mathcal{O}_Z[-1] \rightarrow L \otimes I_Z \xrightarrow{\xi} \mathcal{O}_X[2]$$

for every inclusion  $k(x) \hookrightarrow \mathcal{O}_Z$ . Given such an inclusion, let  $Z' \subset Z$  be the subscheme given by  $\mathcal{O}_{Z'} \cong \mathcal{O}_Z/k(x)$ . If the composition (10) vanishes, then  $\xi$  factors via  $L \otimes I_Z \hookrightarrow L \otimes I_{Z'}$ . This contradicts our assumption  $\text{Ext}^2(L \otimes I_{Z'}, \mathcal{O}_X) = H^1(X, L \otimes I_{Z'} \otimes K_X)^\vee = 0$ .

Now we apply  $\mathbb{D}_L$  to the exact triangle  $\mathcal{O}_X[1] \rightarrow F \rightarrow \mathcal{O}_Z[-1]$ . As  $\mathbb{D}_L(\mathcal{O}_X[1]) = L$  and  $\mathbb{D}_L(\mathcal{O}_Z[-1]) = \mathcal{O}_Z[-1]$ , dualizing (9) gives an exact triangle  $\mathcal{O}_Z[-1] \rightarrow \mathbb{D}_L(F) \rightarrow L \rightarrow \mathcal{O}_Z$ . Since  $\text{Hom}(\mathbb{D}_L(F), k(x)[-1]) = \text{Hom}(k(x)[-1], F) = 0$  for all  $x \in X$ , the map  $L \rightarrow \mathcal{O}_Z$  must be surjective, and hence  $\mathbb{D}_L(F) \cong L \otimes I_Z$ . Consequently, applying  $\mathbb{D}_L$  to the exact triangle  $F \rightarrow E \rightarrow L$  shows that  $\mathbb{D}_L(E)$  is indeed a non-trivial extension of the form (6).  $\square$

**3.2. Chern classes of destabilizing subobjects.** By Proposition 3.1 and Proposition 2.5, Conjecture 2.3 implies the existence of  $t_0 \in (0, \frac{1}{8}]$  with the following properties:

- $E$  is  $\nu_{t_0}$ -semistable.
- There exists an exact sequence in  $\mathcal{B}$

$$(11) \quad 0 \rightarrow A \rightarrow E \rightarrow F \rightarrow 0,$$

with  $\nu_t(A) > 0$  if  $t < t_0$ , and  $\nu_{t_0}(A) = 0$ .

In the remainder of this section, we will prove the following statement:

**Proposition 3.3.** *Assume that  $X, L, \alpha$  satisfy Assumptions (A) and (B) of Theorem 4.1 and Assumption (\*) of the previous section. Then in any destabilizing sequence (11), the object  $A$  is of the form  $L \otimes I_C$ , for some purely one-dimensional subscheme  $C \subset X$  containing  $Z$ .*

We will first prove this for subobjects satisfying  $L^2 \cdot \text{ch}_1^{L/2}(A) \leq L^2 \cdot \text{ch}_1^{L/2}(F)$ , or, equivalently,

$$(12) \quad L^2 \cdot \text{ch}_1^{L/2}(A) \leq \frac{1}{2} L^2 \cdot \text{ch}_1^{L/2}(E) = \frac{d}{2}.$$

(We will later use the derived duality  $\mathbb{D}_L(\_)$  to reduce to this case.)

**Lemma 3.4.** *Any subobject  $A$  satisfying (12) is a sheaf with  $\text{rk}(A) = \text{rk}(H^0(A)) > 0$ .*

*Proof.* Consider the long exact cohomology sequence for  $A \hookrightarrow E \twoheadrightarrow F$ . If  $H^{-1}(A) \neq 0$ , then  $H^{-1}(A) \hookrightarrow \mathcal{O}_X$  is isomorphic to an ideal sheaf of some subscheme  $Y$  of  $X$ . Since  $\mathcal{O}_Y \hookrightarrow H^{-1}(F)$  and  $H^{-1}(F)$  is torsion-free, we must have  $H^{-1}(A) \cong \mathcal{O}_X$ . Then  $H^0(A)$  is also torsion-free, and (12) implies

$$L^2 \cdot \text{ch}_1^{L/2}(H^0(A)) = L^2 \cdot \text{ch}_1^{L/2}(A) - L^2 \cdot \text{ch}_1^{L/2}(\mathcal{O}_X[1]) \leq \frac{d}{2} - \frac{d}{2} = 0.$$

On the other hand, by construction of  $\mathcal{B}$ , every HN-filtration factor  $U$  of  $H^0(A)$  satisfies  $L^2 \cdot \text{ch}_1^{L/2}(U) > 0$ ; thus  $H^0(A) = 0$  and  $A = \mathcal{O}_X[1]$ . This contradiction proves  $H^{-1}(A) = 0$ .

Finally, note that if  $A = H^0(A)$  is a torsion-sheaf, then  $\nu_t(A)$  is independent of  $t$ , again a contradiction.  $\square$

**Lemma 3.5.** *Either  $A$  is torsion-free, or its torsion-part  $A_t$  satisfies*

$$L^2 \cdot \text{ch}_1(A_t) - 2L \cdot \text{ch}_2(A_t) \geq 0 \quad \text{and} \quad L^2 \cdot \text{ch}_1(A_t) > 0.$$

*Proof.* The sheaf  $A_t$  is a subobject of  $E$  in  $\mathcal{B}$  with  $\text{rk} = 0$ . Hence  $L \cdot \text{ch}_2^{L/2}(A_t) \leq 0$ , otherwise it would destabilize  $E$  at  $t = \frac{1}{8}$ . Expanding  $\text{ch}_2^{L/2}$  gives the first inequality. To show the second inequality, we just observe that there are no non-trivial morphisms from sheaves supported in dimension  $\leq 1$  to  $E$ .  $\square$

**Lemma 3.6.** *In the HN-filtration of  $A$  with respect to slope-stability, there exists a factor  $U$  of rank  $r$  such that  $\Gamma := L - \frac{\text{ch}_1(U)}{r}$  satisfies the following inequalities:*

$$(I) \quad L^2 \cdot \Gamma \leq L \cdot \Gamma^2 + 6\alpha$$

$$(II) \quad \frac{d}{2} \left( 1 - \frac{1}{r} \right) \leq L^2 \cdot \Gamma < \frac{d}{2}.$$

*The case  $r = 1$  and  $L^2 \cdot \Gamma = 0$  only occurs when  $A$  is a torsion-free sheaf of rank one and  $H^{-1}(F) = \mathcal{O}_X$ .*

If  $A$  was a line bundle, the above definition of  $\Gamma$  would be just as Reider's original argument for surfaces: in this case,  $\Gamma$  is the support of the cokernel of  $A \hookrightarrow H^0(E) \cong L \otimes I_Z$ .

*Proof.* From  $\nu_{t_0}(A) = 0$  we obtain

$$(13) \quad t_0 = \frac{L \cdot \text{ch}_2^{L/2}(A)}{\text{rk}(A)d}.$$

Applying the conjectured inequality (5) to  $E$ , and plugging in  $t_0$  gives

$$\frac{d}{24} - \alpha = \text{ch}_3^{L/2}(E) \leq \frac{L^2 \cdot \text{ch}_1^{L/2}(E)}{3} t_0 = \frac{d}{3} \frac{L \cdot \text{ch}_2^{L/2}(A)}{\text{rk}(A)d} = \frac{1}{3} \frac{L \cdot \text{ch}_2^{L/2}(A)}{\text{rk}(A)}.$$

We want to bound  $L \cdot \text{ch}_2^{L/2}(A)$ . First we expand  $\text{ch}_2^{L/2}(A)$ :

$$\text{ch}_2^{L/2}(A) = \text{ch}_2(A) - \frac{L \cdot \text{ch}_1(A)}{2} + \text{rk}(A) \frac{L^2}{8}.$$

Substituting, we deduce

$$(14) \quad \frac{L^2 \cdot \text{ch}_1(A)}{\text{rk}(A)} - 2 \frac{L \cdot \text{ch}_2(A)}{\text{rk}(A)} \leq 6\alpha.$$

Let  $A_{tf}$  denote the torsion-free part of  $A$ , and consider its HN-filtration. Among the HN factors, we choose a torsion-free sheaf  $U$  for which the function

$$\eta(\_) := \frac{L^2 \cdot \text{ch}_1(\_) - 2L \cdot \text{ch}_2(\_)}{\text{rk}(\_)}$$

is minimal. Notice that  $\eta$  satisfies the see-saw property: for an exact sequence of torsion-free sheaves

$$0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0,$$

we have  $\eta(N) \geq \min\{\eta(M), \eta(P)\}$ . Hence we get a chain of inequalities leading to

$$(15) \quad \eta(U) \leq \eta(A_{tf}) \leq \eta(A) \leq 6\alpha$$

where we used Lemma 3.5 for the second inequality.

To abbreviate, we now write  $D := \text{ch}_1(U)$  and  $r := \text{rk}(U)$ . Since  $U$  is  $\mu_{\omega, L/2}$ -semistable, we can combine the classical Bogomolov-Gieseker inequality with (15) to obtain

$$L^2 \cdot \frac{D}{r} = \frac{2L \cdot \text{ch}_2(U)}{r} + \eta(U) \leq L \cdot \frac{D^2}{r^2} + 6\alpha.$$

Substituting  $D = rL - r\Gamma$  yields the inequality (I).

To prove the chain of inequalities (II), we observe on the one hand that  $L^2 \cdot \text{ch}_1^{L/2}(U) > 0$  by the definition of  $\mathcal{T}_{\omega,B} = \mathcal{B} \cap \text{Coh } X$ . On the other hand,  $U$  is a subquotient of  $A$  in  $\mathcal{T}_{\omega,B}$ ; combined with inequality (12) we obtain

$$0 < L^2 \cdot \text{ch}_1^{L/2}(U) \leq L^2 \cdot \text{ch}_1^{L/2}(A) \leq \frac{d}{2}.$$

Plugging in  $\text{ch}_1^{L/2}(U) = -\frac{r}{2}L + D = \frac{r}{2}L - r\Gamma$  shows the inequality (II).

Finally, note that in the case  $r = 1$  and  $L^2 \cdot \Gamma = 0$  the chain of inequalities leading to the first part of (II) must be equalities; in particular  $L^2 \cdot \text{ch}_1^{L/2}(U) = L^2 \cdot \text{ch}_1^{L/2}(A)$ . This shows that  $A_{tf}$  cannot have any other HN-filtration factors besides  $U$ , i.e.,  $U = A_{tf}$ . Additionally it implies that  $\text{ch}_1^{L/2}(A_t) = 0$ , in contradiction to Lemma 3.5; hence  $A_t = 0$  and  $A = U$  is a torsion-free rank one sheaf.

As  $L \otimes I_Z$  is torsion-free, if the image of  $H^{-1}(F) \rightarrow A$  is non-trivial, then the map is surjective, and the inclusion  $A \hookrightarrow E$  factors via  $A \hookrightarrow \mathcal{O}_X[1] \hookrightarrow E$ , in contradiction to the stability of  $\mathcal{O}_X[1]$  for all  $t$  and  $\nu_{t_0-\varepsilon}(A) > 0 > \nu_{t_0-\varepsilon}(\mathcal{O}_X[1])$ . Thus  $H^{-1}(F) = \mathcal{O}_X$ .  $\square$

*Proof.* (Proposition 3.3) We combine (I) and (II) with the Hodge Index Theorem (just as in [AB11, Corollary 3.9]) to obtain

$$(L \cdot \Gamma^2) d \leq (L^2 \cdot \Gamma)^2 \leq \frac{d}{2} (L \cdot \Gamma^2 + 6\alpha),$$

and so  $L \cdot \Gamma^2 \leq 6\alpha$ .

In the case  $r > 1$ , we use (I) and (II) again to get

$$\frac{d}{4} \leq L^2 \cdot \Gamma \leq L \cdot \Gamma^2 + 6\alpha \leq 12\alpha,$$

and so  $d \leq 48\alpha$  in contradiction to Assumption (A).

Reider's original argument in [Rei88] deals with the case  $r = 1$ : In case  $L^2 \cdot \Gamma \neq 0$ , then  $L^2 \cdot \Gamma \geq 1$ . Let  $\kappa := L \cdot \Gamma^2 \leq 6\alpha$ . Again combining the Hodge Index Theorem with (I), we obtain

$$(L \cdot \Gamma^2) d \leq (L \cdot \Gamma^2 + 6\alpha)^2,$$

and so

$$d \leq 12\alpha + \frac{\kappa^2 + 36\alpha^2}{\kappa}.$$

The RHS is strictly decreasing function for  $\kappa \in (0, 6\alpha]$  and equals  $49\alpha$  for  $\kappa = \alpha$ ; thus Assumption (A) implies  $\kappa < \alpha$ . On the other hand,  $\Gamma$  is integral, and hence Assumption (B) implies  $L^2 \cdot \Gamma \geq 7\alpha$ , in contradiction to (I).

Finally, if  $L^2 \cdot \Gamma = 0$ ; then, according to Lemma 3.6, we have  $H^{-1}(F) \cong \mathcal{O}_X$ . Hence  $A$  is a subsheaf of  $L \otimes I_Z$  with  $\text{ch}_1(A) = \text{ch}_1(L)$ ; this is only possible if  $A \cong L \otimes I_W$ , for some closed subscheme  $W \subset X$  with  $\dim(W) \leq 1$ . If  $W$  is zero-dimensional, then  $\text{ch}_2^{L/2}(A) = \frac{1}{2}L^2$  and equation (13) gives  $t_0 = \frac{1}{2}$ , in contradiction to  $t_0 \in (0, \frac{1}{8}]$ . Hence  $W$

is one-dimensional, and we have shown that any subobject  $A$  with  $\mathrm{ch}_1^{L/2}(A) \leq \frac{d}{2}$  is of the form  $A \cong L \otimes I_W$ . In particular  $\mathrm{ch}_1^{L/2}(A) = \frac{d}{2}$  in this case, so there are no subobject with  $\mathrm{ch}_1^{L/2}(A) < \frac{d}{2}$ .

Now assume  $\mathrm{ch}_1^{L/2}(A) > \frac{d}{2}$ . We can apply Proposition 3.2 and Proposition 2.7 to the short exact sequence (11) obtain a short exact sequence in  $\mathcal{B}$

$$0 \rightarrow \tilde{F} \xrightarrow{u} E \rightarrow E/\tilde{F} \rightarrow 0$$

which is again destabilizing. Indeed, since  $\mathcal{B}$  is the heart of a bounded t-structure, there exists a cohomology functor  $H_{\mathcal{B}}^*(\_)$ . Applied to the exact triangle

$$\mathbb{D}_L(F) \rightarrow \mathbb{D}_L(E) = E \rightarrow \mathbb{D}_L(A),$$

it induces a long exact sequence in  $\mathcal{B}$

$$(16) \quad 0 \rightarrow \tilde{F} = H_{\mathcal{B}}^0(\mathbb{D}_L(F)) \xrightarrow{u} E \rightarrow \tilde{A} \rightarrow T_0 = H_{\mathcal{B}}^1(\mathbb{D}_L(F)) \rightarrow 0.$$

As  $\mathbb{D}_L$  preserves  $L^2 \cdot \mathrm{ch}_1^{L/2}(\_)$ , we have that  $\tilde{F}$  is a destabilizing subobject with  $\mathrm{ch}_1^{L/2}(F) = \mathrm{ch}_1^{L/2}(E) - \mathrm{ch}_1^{L/2}(A) < \frac{d}{2}$ , which does not exist.

Finally, note that the long exact sequence (16) also implies that  $\mathbb{D}_L(A) = \tilde{A} \in \mathcal{B}$ . This gives the vanishing of  $0 = \mathrm{Hom}(\mathbb{D}_L(A), k(x)[-1]) = \mathrm{Hom}(k(x)[-1], A)$ . This is equivalent to the claim that  $W$  is a purely one-dimensional scheme, as any subsheaf  $k(x) \hookrightarrow \mathcal{O}_W$  gives an extension of  $k(x)$  by  $L \otimes I_W$ . This finishes the proof of Proposition 3.3.  $\square$

#### 4. A REIDER-TYPE THEOREM

In this section we prove our main theorem:

**Theorem 4.1.** *Let  $L$  be an ample line bundle on a smooth projective threefold  $X$ , and assume Conjecture 2.3 holds for  $B$  and  $\omega$  proportional to  $L$ . Fix a positive integer  $\alpha$ , and assume that  $L$  satisfies the following conditions:*

- (A)  $L^3 > 49\alpha$ ;
- (B)  $L^2 \cdot D \geq 7\alpha$ , for all integral divisor classes  $D$  with  $L^2 \cdot D > 0$  and  $L \cdot D^2 < \alpha$ ;
- (C)  $L \cdot C \geq 3\alpha$ , for all curves  $C$ .

*Then  $H^1(X, K_X \otimes L \otimes I_Z) = 0$ , for any zero-dimensional subscheme  $Z \subset X$  of length  $\alpha$ .*

*Proof.* As explained in Section 3.1, we may proceed by induction on the length of  $Z$  and may use Assumption (\*). Let  $t_0 \in (0, \frac{1}{8}]$  be as in Section 3.2 and let  $t = t_0 - \epsilon$ . Truncating the Harder-Narasimhan filtration of  $E$  with respect to  $\nu_t$ -stability gives a short exact sequence

$$0 \rightarrow A \rightarrow E \rightarrow F \rightarrow 0$$

with  $\nu_t(A) > 0$ , such that any subobject  $A' \hookrightarrow E$  with  $\nu_t(A') > 0$  factors via  $A' \hookrightarrow A$ . By Proposition 3.3,  $A$  is of the form  $L \otimes I_C$  for some purely one-dimensional subscheme  $C \subset X$ ; it also implies that  $A$  is stable, as any destabilizing subobject  $A'$  of  $A$  would again be of the form  $A' \cong L \otimes I_{C'}$ , so that the quotient  $A/A'$  would be a torsion sheaf with  $\nu_t(A/A') = +\infty$ .

Let  $\tilde{F}$  be the object obtained by dualizing  $F$  and applying Proposition 2.7. The map  $\mathbb{D}_L(F) \rightarrow \mathbb{D}_L(E) \cong E$  induces a map  $\tilde{F} \rightarrow E$  which is an injection in  $\mathcal{B}$ . Since

$$(17) \quad \mathrm{ch}_i^{L/2}(\tilde{F}) = \mathrm{ch}_i^{L/2}(\mathbb{D}_L(F))$$

for  $i \leq 2$ , we have  $\nu_t(\tilde{F}) = -\nu_t(F) > 0$ ; thus the map factorizes as  $\tilde{F} \hookrightarrow A \hookrightarrow E$ . By Proposition 3.3, the object  $\tilde{F}$  is of the form  $L \otimes I_{C'}$  for some purely one-dimensional subscheme  $C' \subset X$ . Equation (17) also implies  $\mathrm{ch}_i^{L/2}(\tilde{F}) = \mathrm{ch}_i^{L/2}(A)$  for  $i \leq 2$ ; thus the (non-trivial) map  $L \otimes I_{C'} \rightarrow L \otimes I_C$  has zero-dimensional cokernel. It follows that

$$\mathrm{ch}_3^{L/2}(F) = \mathrm{ch}_3^{L/2}(\mathbb{D}_L(F)) \leq \mathrm{ch}_3^{L/2}(\tilde{F}) \leq \mathrm{ch}_3^{L/2}(A).$$

This implies that

$$(18) \quad 2 \mathrm{ch}_3^{L/2}(A) \geq \mathrm{ch}_3^{L/2}(A) + \mathrm{ch}_3^{L/2}(F) = \mathrm{ch}_3^{L/2}(E) = \frac{d}{24} - \alpha,$$

and the difference of the two sides is a non-negative integer.

On the other hand, as  $A$  is stable, by Conjecture 2.3, by (13) and (18), and by expanding  $\mathrm{ch}^{L/2}$  we have

$$(19) \quad \frac{d}{48} - \frac{\alpha}{2} \leq \mathrm{ch}_3^{L/2}(A) \leq \frac{t_0}{3} L^2 \cdot \mathrm{ch}_1^{L/2}(A) = \frac{1}{6} L \cdot \mathrm{ch}_2^{L/2}(A) = \frac{d}{48} - \frac{L.C}{6}.$$

We now use Assumption (C):  $L.C \geq 3\alpha$ . This contradicts (19), unless  $L.C = 3\alpha$  and

$$\frac{d}{48} - \frac{\alpha}{2} = \mathrm{ch}_3^{L/2}(A) = \frac{t_0}{3} L^2 \cdot \mathrm{ch}_1^{L/2}(A).$$

Since  $(TL) \cdot \Delta(A) = 3\alpha T \neq 0$ , this in turn contradicts Proposition 2.6.  $\square$

We also obtain the following result characterizing the only possible counter-examples to Fujita's very ampleness conjecture in case  $L = M^5$ :

**Proposition 4.2.** *Assume that Conjecture 2.3 holds for  $X$ ,  $\omega = tL$  and  $B = \frac{L}{2}$  and  $L \cong M^5$  for an ample line bundle  $M$ . Then either  $K_X \otimes L$  is very ample, or there exists a curve  $C$  of degree  $M.C = 1$  and arithmetic genus  $g_a(C) = \frac{5}{2} + \frac{1}{2}K_X.C$  such that  $K_X \otimes L|_C$  is a line bundle of degree  $2g_a(C)$  on  $C$  which is not very ample.*

*Proof.* Assume that  $K_X \otimes L$  is not very ample. We follow the logic and the notation of the proof of Theorem 4.1, with  $\alpha = 2$ . As before, let  $A = L \otimes I_C$  be the destabilizing subobject of  $E$  for  $t = t_0 - \epsilon$ ; here  $C$  is a purely one-dimensional subscheme of  $X$ . By the proof of Theorem 4.1, we have  $L.C < 6$  and thus necessarily  $M.C = 1$  and  $L.C = 5$ . In

particular,  $C$  is reduced and irreducible. We claim that  $\text{ch}_3^{L/2}(A) = \frac{d}{48} - 1$ . Indeed, setting  $\alpha = 2$  in (19) gives

$$(20) \quad \frac{d}{48} - 1 \leq \text{ch}_3^{L/2}(A) \leq \frac{d}{48} - \frac{5}{6}.$$

On the other hand, if  $\text{ch}_3^{L/2}(A) \neq \frac{d}{48} - 1$ , then, by (18),  $\text{ch}_3^{L/2}(A) \geq \frac{d}{48} - \frac{1}{2}$ , a contradiction to the inequality (20).

From the claim, we obtain

$$\text{ch}_3(L \otimes \mathcal{O}_C) = \text{ch}_3(L) - \text{ch}_3(A) = \frac{7}{2}$$

and thus

$$\text{ch}_3(\mathcal{O}_C) = \text{ch}_3(L \otimes \mathcal{O}_C) - L.C = -\frac{3}{2}$$

By Hirzebruch-Riemann-Roch, we get

$$1 - g_a(C) = \text{ch}_3(\mathcal{O}_C) - \frac{1}{2}K_X.C.$$

Plugging in the previous equation and solving for  $K_X.C$  shows that  $K_X \otimes L|_C$  is a line bundle of degree  $2g_a(C)$  on  $C$ . The explicit expression for  $g_a(C)$  follows immediately.

Finally, the cohomology sheaves of the quotient  $F \cong E/A$  are  $H^{-1}(F) \cong \mathcal{O}_X$  and  $H^0(F) \cong L \otimes \mathcal{O}_C(-Z)$  (where  $\mathcal{O}_C(-Z)$  denotes the ideal sheaf of  $Z \subset C$ ). If  $F$  were decomposable,  $\tilde{F}$  would be a decomposable destabilizing subobject of  $E$ , which cannot exist. Hence

$$0 \neq \text{Ext}^2(L \otimes \mathcal{O}_C(-Z), \mathcal{O}_X) = H^1(C, K_X \otimes L|_C(-Z))^\vee.$$

On the other hand,  $K_X \otimes L|_C$  is a line bundle of degree  $2g_a(C)$  on an irreducible Cohen-Macaulay curve, and thus  $H^1(K_X \otimes L|_C) = 0$ . Hence  $K_X \otimes L|_C$  is not very ample.  $\square$

**Remark 4.3.** Notice that Proposition 4.2 implies Fujita's conjecture when  $K_X$  is numerically trivial (or, more generally, when  $K_X.C$  is even for all integral curve classes  $C$ ).

In case the curve  $C \subset X$  of Proposition 4.2 is l.c.i, one can be even more precise. Let  $\omega_C$  be the dualizing sheaf (which agrees with the dualizing complex, as  $\mathcal{O}_C$  is pure and thus  $C$  Cohen-Macaulay). The sheaf  $K_X \otimes L(-Z)|_C$  is torsion-free of rank one and degree  $2g_a(C) - 2$  with  $H^1(K_X \otimes L(-Z)|_C) \neq 0$ , and thus Serre duality implies  $K_X \otimes L(-Z)|_C \cong \omega_C$ . If  $N$  is the normal bundle, adjunction gives  $\Lambda^2 N \cong L(-Z)$ . In particular, the normal bundle has degree 3. Since  $M.C = 1$ , bend-and-break implies that such a curve cannot be rational.

In conclusion, we show how to reverse part of the argument in this section when  $Z$  has length one. Indeed, in such a case we can use Ein-Lazarsfeld theorem (or better, its variant



by Kawamata and Helmke) to show that Conjecture 2.3 holds true for this particular case, coherently with our result:

**Proposition 4.4.** *Let  $L$  be an ample line bundle on a smooth projective threefold  $X$ . Assume that  $L$  satisfies the following conditions:*

- (a)  $L^3 \geq 28$ ;
- (b)  $L^2 \cdot D \geq 9$ , for all integral effective divisor classes  $D$ .

*Assume also that there exists  $x \in X$  such that  $H^1(X, K_X \otimes L \otimes I_x) \neq 0$ . Then Conjecture 2.3 holds for all objects  $E \in \mathcal{B}$  given as non-trivial extensions*

$$\mathcal{O}_X[1] \rightarrow E \rightarrow L \otimes I_x \rightarrow \mathcal{O}_X[2].$$

*Proof.* The argument is very similar to [Kaw97], Proposition 2.7 and Theorem 3.1, Step 1. We freely use the notation from [Laz04, Sections 9 & 10]. By [Kaw97, Lemma 2.1], given a rational number  $t$  satisfying  $3/\sqrt[3]{L^3} < t < 1$ , there exists a  $\mathbb{Q}$ -divisor  $D$  numerically equivalent to  $tL$  such that  $\text{ord}_x D = 3$ . Let  $c \leq 1$  the log-canonical threshold of  $D$  at  $x$ .

By [Kaw97, Theorem 3.1] (also [Hel97]) and our assumptions, the LC-locus  $\text{LC}(cD; x)$  (i.e., the zero-locus of the multiplier ideal  $\mathcal{J}(c \cdot D)$  passing through  $x$ ) must be a curve  $C$  satisfying  $1 \leq L \cdot C \leq 2$ . We can now apply Nadel's vanishing theorem to  $cD$  to deduce that  $H^1(X, K_X \otimes L \otimes I_C) = 0$ , and so that the restriction map  $H^0(X, K_X \otimes L) \rightarrow H^0(X, K_X \otimes L|_C)$  is surjective.

Consider the composition  $u: L \otimes I_C \rightarrow L \otimes I_x \rightarrow \mathcal{O}_x[2]$ . Then,  $u \neq 0$  if and only if  $x$  is a base point of  $K_X \otimes L$  which is not a base point of  $K_X \otimes L|_C$ . The surjectivity of the restriction map implies that  $u = 0$ . Hence, we get an inclusion  $L \otimes I_C \hookrightarrow E$  in  $\mathcal{B}$  which destabilizes  $E$ , if (2) is not satisfied.  $\square$

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